A relative gradient theory for layered materials

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Abstract. It is possible to remedy certain difficulties with the description of short wave length phenomena and interfacial slip in standard models of a laminated material by considering the bending stiffness of the layers. If the couple or moment stresses are assumed to be proportional to the relative deformation gradient, then the bending effect disappears for vanishing interface slip, and the model correctly reduces to an isotropic standard continuum. In earlier Cosserat-type models this was not the case. Laminated materials of the kind considered here occur naturally as layered rock, or at a different scale, in synthetic layered materials and composites. Similarities to the situation in regular dislocation structures with couple stresses, also make these ideas relevant to single slip in crystalline materials. Application of the theory to a one-dimensional model for layered beams demonstrates agreement with exact results at the extremes of zero and infinite interface stiffness. Moreover, comparison with finite element calculations confirm the accuracy of the prediction for intermediate interfacial stiffness.

1. INTRODUCTION

In a Cosserat Continuum [6], microelement rotations which are generally different from the local rotations of the continuum are introduced, together with associated couple stresses. The Cosserat Continuum Theory (CCT) has been applied successfully in the analysis of materials composed of elastic layers with alternating elastic coefficients. For example Biot [1] used a CCT with constrained rotations while Mühlhaus [2, 3] allowed free rotations. However, difficulties are encountered with the proper representation of certain limit cases. In the following, a brief outline of a model is given which is derived within the framework of a relative gradient theory [4]. The theory is a generalisation of the CCT and related theories and remedies some of their shortcomings.

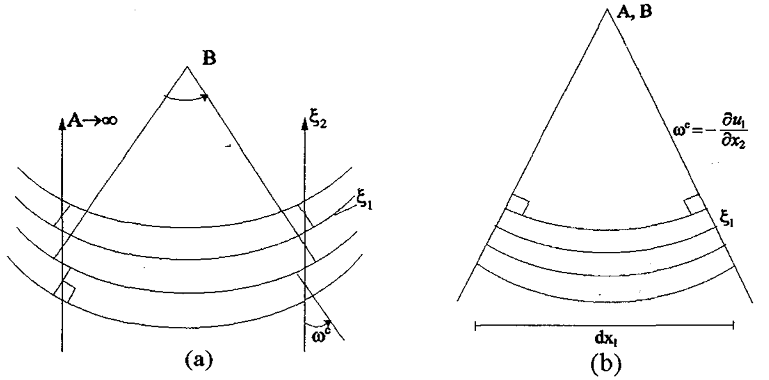

Figure 1. Bending of a layered material

The basic ideas are explained by considering the example in Fig. 1a and 1b. In Fig. 1a it is assumed that the layers are perfectly smooth, that is, the shear stiffness of the interface vanishes. We identify the

Cosserat rotation ω^c with the cross-sectional rotation of the layers. In this case there is no choice for the couple stress-curvature relation other than

$$m = \frac{Eh^2}{12(1-\nu^2)}\kappa, \qquad \kappa = \frac{\partial \omega^c}{\partial x_1} \tag{1}$$

where h is the layer thickness. In Fig. 1b we consider continuous pure bending of the layered system. For pure bending, continuity of the deformation across the interfaces would seem to require that

$$\omega^c = -\frac{\partial u_1}{\partial x_2} \quad . \tag{2}$$

Couple stresses should not be present in this case, since the term in (2) is resisted by the usual stresses generated in an elastic solid. However, equation (2) contradicts equation (1) according to which $m \neq 0$. However, a modified definition for the moment generating part of the curvature reconciles the two cases. We assume as in (1), that

$$m = \frac{Eh^2}{12(1-\nu^2)}\kappa^{rel}, \quad \text{but } \kappa^{rel} = \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2} + \omega^c\right). \tag{3}$$

When the derivative of u_1 with respect to x_2 vanishes as in Fig. 1a then $\kappa^{rel} = \kappa$. In this case the definitions (1) and (3) coincide. In pure bending of the layered system as in Fig. 1b equation (2) holds and accordingly $\kappa^{rel} = 0$ and $m = 0$. Motivated by this simple model we shall derive a consistent continuum theory for layered materials. It is understood that the assumptions in (3) are purely *ad hoc* and therefore constitutive by nature. However, this is not to say that (3) is without meaning or motivation. Normal elasticity will deal with the deformational spin (the antisymmetric part of the displacement gradient) without modification. However, in the case of Figure 1a, we have a situation where the microstructural spin of the layer cross-sections is different from the spin of the associated deformation field. The difference between these two spins is precisely the slip across the interfaces, and κ^{rel} is the gradient of this interfacial slip generated relative spin. Such gradients of relative spin arise as the natural generators of microscopic moment stresses in theories of granular materials [3], and amount to corrections to elasticity required by systems that generate moments in the process of violating continuity. However, the moment generating relative slip is often more isotropic in a general granular theory, since there is interfacial slip occurring (in the statistical sense) in all orientations.

2. FIELD EQUATIONS

As discussed in the definition of the layer bending moment we use the relative curvature

$$\kappa^{rel} = \gamma_{12,1} = \kappa_{121} \quad \text{where} \quad \gamma_{12} = u_{1,2} + \omega_3^c . \tag{6}$$

The relative curvature κ_{121} is a component of the third order tensor

$$\kappa_{ijk} = \gamma_{ij,k} \quad \text{with} \quad \gamma_{ij} = u_{i,j} + e_{ijk}\omega_k^c , \tag{7}$$

and the antisymmetric part of γ is the relative spin, while the symmetric part is the strain. We now derive the field equations for a medium with deformation described by γ_{ij} and κ_{ijk}. This resulting continuum is neither a Cosserat nor a Mindlin [5] medium, both of which consider gradients of the microstructural freedoms rather than gradients of the relative motions.

We write the virtual work principle for independent variations δu_i and $\delta\omega_3^c$ as

$$\delta W - \delta W_{ext} = 0 , \tag{8}$$

where

$$\delta W = \int_A \left(\sigma_{ij}\delta\gamma_{ij} + m_{ijk}\delta\kappa_{ijk}\right) da , \tag{9}$$

Here, σ_{ij} and m_{ijk} designate the stress and couple stress tensor respectively and δW_{ext} is the virtual work done by the external forces. The field equations are then obtained in the usual way by successive application of the divergence theorem to the right-hand side of (9). The surface terms associated with the application of Gauss' theorem are

$$\int_S \tau_{ij} n_j \delta u_i dA + \int_S m_{ijk} n_k \delta\gamma_{ij} dA \tag{10}$$

where we have introduced the stress

$$\tau_{ij} = \sigma_{ij} - m_{ijk,k} \tag{11}$$

and have extended our consideration to the case of a 2-D surface S bounding a 3-D volume.

From the surface integrals appearing in (10) we can deduce the boundary conditions necessary for a complete specification of the mechanical problem. The first of these terms already involves **u** directly, and hence will cause no special difficulty. The last term can be written as

$$\int_S m_{ijk} n_k \delta\gamma_{ij} dA = \int_S m_{ijk} n_k \delta u_{i,j} dA + \int_S m_{ijk} n_k e_{ijn} \delta\omega_n dA \tag{12}$$

and again, the second term on the RHS of (12), involving **ω** directly, does not cause any difficulty. However, the remaining term is considerably less accommodating, due to the fact that tangent derivatives of **u** are determined by specifying **u** on the boundary, while the normal derivatives must be specified separately. Thus the first term on the RHS of (12) must be decomposed into two parts, and the part determined by **u** must be written so that the integrand contains **u** explicitly. For brevity, define $\phi_{ij} = m_{ijk} n_k$. Then the integrand of the term of interest is

$$\begin{aligned} \phi_{ij}\delta u_{i,j} &= \phi_{ij}\left(\delta u_{i,j} - \delta u_{i,m} n_m n_j\right) + \phi_{ij}\delta u_{i,m} n_m n_j \\ &= \left[\partial_j - n_j n_m \partial_m\right]\phi_{ij}\delta u_i - \left(\phi_{ij,j} - \phi_{ij,m} n_m n_j\right)\delta u_i + \phi_{ij}\delta u_{i,m} n_m n_j \end{aligned} \tag{13}$$

The surface divergence theorem

$$\int_S \left[\partial_j - n_j n_m \partial_m\right] a_j \, dA = \int_S a_m n_m \left[\partial_j - n_j n_m \partial_m\right] n_j \, dA \tag{14}$$

follows from decomposing the vector **a** into tangent and normal parts, and noting that the integral of the surface divergence of the tangent part vanishes. (That S is closed follows from the fact that it bounds a volume, and we also assume it to be smooth). Applying the chain rule to the remaining normal part shows that one of the terms in the chain rule vanishes identically, leading to (14). Putting all this together leads to the three essential and corresponding natural boundary condition pairs

$$\begin{aligned} & m_{ijk} n_j n_k \quad \text{or} \quad u_{i,j} n_j \\ & m_{ijk} n_k e_{ijn} \quad \text{or} \quad \omega_n \\ & \tau_{ij} n_j + m_{ijk,n} n_j n_k n_n + \left[\left(n_{n,n} - n_{n,m} n_n n_m\right) m_{ijk} + m_{ijm} n_{m,k}\right] n_j n_k \\ & - m_{ijk,j} n_k - m_{ijk} n_{k,j} \quad \text{or} \quad u_i \end{aligned} \tag{15}$$

The deformation measures γ_{ij} and κ_{ijk} are valid only when the deformation is infinitesimal. For the treatment of large deformation problems we require an appropriate generalisation of the relative deformation tensor γ_{ij}. Perhaps the most straight forward generalisation reads [7] $\Gamma = (\mathbf{R}^c)^T \mathbf{F}$, where **F** is the deformation gradient and $\mathbf{R}^c$ is the rotation tensor of the Cosserat triad. When the Cosserat rotation is equal to the rotation **R** of an infinitesimal element of the continuum, then it follows from the polar decomposition **F** = **RU** that Γ = **U**, where **U** is the right Cauchy-Green tensor. For infinitesimal deformation Γ_{ij} reduces to γ_{ij}. The starting point for the derivation of a large deformation theory is again the virtual work principle (8) and (9) in which we replace $\delta\gamma_{ij}$ and $\delta\kappa_{ijk} = \delta\gamma_{ij,k}$ by $\delta\Gamma_{ij}$ and $\delta\Gamma_{ij,k}$ respectively. Here we are interested in only a relatively simple variant of the theory. In the spirit of the beam and plate buckling theories we assume that the geometric nonlinearity associated with the higher order derivatives $\Gamma_{ij,k}$ and the higher order stresses m_{ijk} are negligible. Then, the incremental form of the virtual work principle is

$$\Delta\delta W = \int_V \Delta\sigma_{ij}\delta\gamma_{ij} dV + \int_V \sigma_{ij}\Delta\delta\Gamma_{ij} dV + \int_V \Delta m_{ijk}\delta\gamma_{ij,k} dV \tag{16}$$

where

$$\Delta\delta\Gamma_{ij} = \delta W^c_{ki}\Delta\gamma_{kj} + \Delta W^c_{ki}\delta u_{k,j} \quad \text{and} \quad W^c_{ki} = -e_{ki3}\omega^c_3 \tag{17}$$

Terms of third order and higher have been dropped, and V designates the domain of integration in the undeformed configuration. Again, the key to the boundary conditions is the trick embodied in (13) and (14), and the derivation is similar to the case of infinitesimal deformations.

3. CONSTITUTIVE RELATIONS

The layer material itself is assumed as isotropic and linear elastic. At the interfaces the layers exhibit two different modes of behaviour: (a) elastically connected with the interface stiffnesses k_{normal} and k_{shear} (k_n and k_s, in short); and b) disconnected with frictional sliding. This type of behaviour is best described by the elasto-plasticity type of relations. With this in mind we decompose the rate of the deformation vector into an elastic and a plastic part. For the interface criterion f and the corresponding plastic potential g we assume that

$$f = |\sigma_{12}| + \mathrm{tg}\varphi\sigma_{22} - c \le 0 \quad \text{and} \quad g = |\sigma_{12}| \tag{18}$$

where φ and c designate the angle of friction and the cohesion of the interface, and x_2 is normal to the layer axis.

The constitutive relations for σ_{11} and σ_{22} are the same as for a conventional continuum and the relationships between the equivalent continuum stiffness and the layer and interface properties are well known. In view of the treatment of nonlinear problems we use an incremental formulation which reads

$$\Delta\sigma_{11} = c_{11}\Delta\gamma_{11} + c_{12}\Delta\gamma_{22}\ ,\ \Delta\sigma_{22} = c_{22}\Delta\gamma_{22} + c_{12}\Delta\gamma_{11}\ , \tag{19}$$

$$c_{11} = \frac{E}{1-\nu^2 - \dfrac{\nu^2(1+\nu)^2}{1-\nu^2+\dfrac{E}{hk_n}}}\ ,\ c_{22} = \frac{1}{\dfrac{1-\nu-2\nu^2}{E(1-\nu)} + \dfrac{1}{hk_n}} \tag{20}$$

$$c_{12} = c_{22}\frac{\nu}{1-\nu}\ , \tag{21}$$

where E and ν are the Young's modulus and the Poisson ratio respectively of the layer material. For a detailed derivation of the above coefficients we refer to Singh [8]. Unlike the situation in a conventional continuum, we have two different equivalent shear moduli. In addition the bending stiffness of the layers is

$$\begin{aligned}\Delta\sigma_{12} &= hk\Delta_s\gamma_{12} = hk_s\left(\Delta u_{1,2} + \Delta\omega_3^c\right),\\ \Delta\sigma_{21} &= G\Delta\gamma_{21} = G\left(\Delta u_{2,1} - \Delta\omega_3^c\right)\end{aligned} \tag{22}$$

and

$$\Delta m_{121} = \frac{Eh^2}{12(1-\nu)^2}\Delta\kappa_{121} = \frac{Eh^2}{12(1-\nu)^2}\Delta\gamma_{12,1} \tag{23}$$

All other components of Δm_{ijk} vanish. In (23), h designates the layer thickness.

4. EXAMPLE

Here we explore the quantitative features of the theory by solving a simple boundary value problem. Consider a simply supported beam under point loading and axial prestress (Fig. 2a). The exact solution of this problem using conventional elasticity theory is prohibitively complex. However, the above formulation does admit a considerably less complex approximate solution in the spirit of a generalised beam theory. The starting point for the derivation of the beam theory is the usual kinematic assumption

$$u_1 = -x_2\Omega(x_1)\ ,\ u_2 = v(x_1) \text{ and } \omega_3^c = \Omega^c(x_1) \tag{24}$$

Inserting (24) into the virtual work principle and integrating over x_2 yields

$$M' + Q = 0\ ,\ m' + Q - T = 0 \text{ and } (Q - \sigma\omega_3^c)' + q = 0 \tag{25}$$

where

$$M = \frac{Et^2}{12(1-\nu^2)}\Omega'\ ,\qquad m = -\frac{Eh^2}{12(1-\nu^2)}(\Omega - \Omega^c)' \tag{26}$$

and

$$Q = G(v' - \Omega^c)\ ,\qquad T = -k_s h(\Omega - \Omega^c)\ ; \tag{27}$$

(the prime designates differentiation with respect to x_1). The forces and moments acting at a macro and micro element of the beam are represented in Fig. 2b. In the derivation of the initial stress term in (25) we have made some additional assumptions (rotations large as compared to strains, for instance) which are typical for all technical beam and plate buckling theories (see [4] for details).

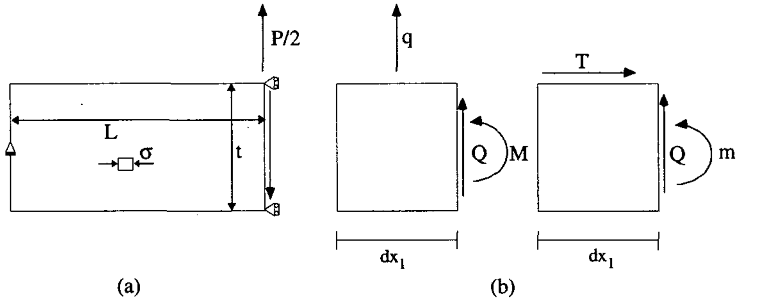

Figure 2. Beam Geometry (a). Forces and moments acting at the macro- and the micro elements (b).

First we assume that $\sigma = q = 0$ initially, then the displacement at the centre of the beam is obtained as

$$v(o) = \frac{P}{2k_s h}\left(\frac{L}{t}\right)\left(1+\frac{k_s h}{G}-\left(\lambda\frac{L}{h}\right)^{-1}\tanh\left(\lambda\frac{L}{h}\right)\right)+\frac{2(1-v^2)}{E}\left(\frac{L}{t}\right)^3 P. \tag{28}$$

By inspection of (28) one concludes that the influence of the micromoments is significant when

$$\tanh\left(\lambda\frac{L}{h}\right) > \frac{1}{10}\left(\lambda\frac{L}{h}\right)\left(1+\frac{k_s h}{G}\right). \tag{29}$$

where

$$\lambda^2 = \frac{12k_s h(1-v^2)}{E} \tag{30}$$

In Fig. 3 v(0) is represented as a function of G/k_sh. Also shown are the results of a Finite Element analysis. In the analysis, 250 eight node isoparametric elements and 4x25 six node joint elements have been used. The difference between the gradient model and the numerical solution is less than 5% in the three cases shown.

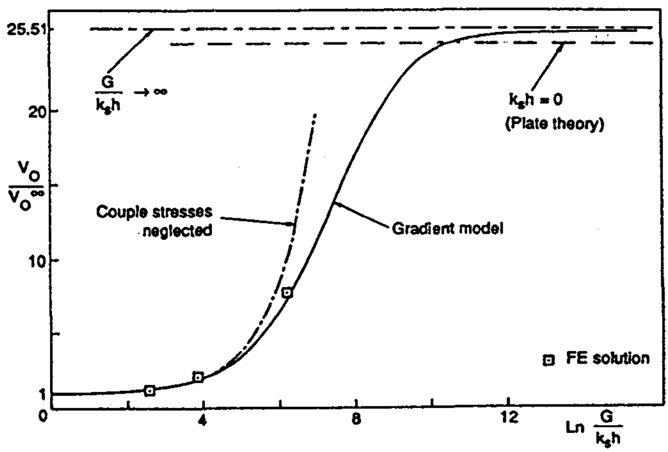

Figure 3. Normalised centre displacement v_o/v_o^∞ as a function of the (log of the) dimensionless shear modulus $\ln(G/k_sh)$, assuming $v = 0$, $t/L = 0.2$ and $h/t = 0.2$; $v_o^\infty = v_o(k_sh \to \infty)$.

Similar results can be derived for the compliance test [9] based upon the above study of the thick laminated beam. In Figure 4, we show the geometry of the compliance test to be considered. Application of the foregoing beam theory leads to the expression for the crack extension force

$$g = P\frac{\partial v(0)}{\partial L} = \frac{P^3}{k_s ht}\left\{1+\frac{k_s h}{G}+\lambda^2\left(\frac{L}{t}\right)^2-\left[\cosh\left(\lambda\frac{L}{h}\right)\right]^{-2}\right\}. \tag{31}$$

As expected from Figure 3, this result generalises the classical solutions, and accords well with them for limiting cases, as is apparent from Figure 5.

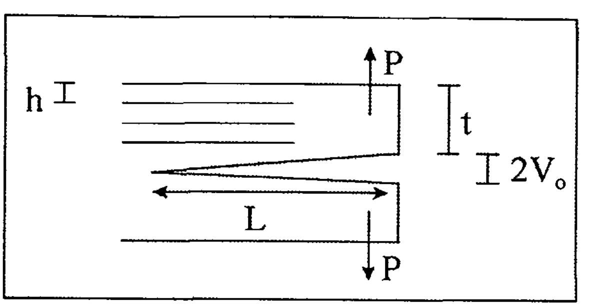

Figure 4. Geometry, and definition of variables for a compliance test.

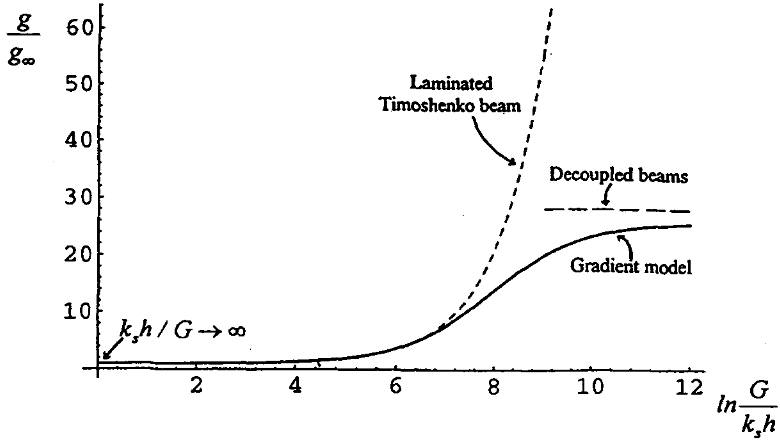

Figure 5. Normalised crack extension force g/g_∞ as a function of the (log of the) dimensionless shear modulus ln(G/k_sh), assuming $\nu = 0$, t/L = 0.2 and h/t = 0.2; $g_\infty = g(k_s h \to \infty)$.

We close this section with the discussion of a simple buckling problem. Here we assume that P = q = 0, and $\sigma > 0$. We are looking for critical values of σ for which the homogenous differential problem admits periodic solutions. Combining (25-27) yields

$$-\frac{Eh^2}{12(1-\nu^2)}\left(\Omega+\frac{(E/\sigma)t^2}{12(1-\nu^2)}\Omega''\right)-\frac{Et^2}{12(1-\nu^2)}\Omega+k_s h\left(\Omega+\frac{(E/\sigma)t^2}{12(1-\nu^2)}\Omega''\right)=0 \tag{32}$$

Substituting

$$\Omega=\sin n\pi\frac{x_1}{2L}\quad,\quad n=1,2,\ldots\ldots.. \tag{33}$$

for the nth buckling mode leads to the critical condition

$$\sigma^*=\frac{\left(\frac{n\pi}{2}\right)^2\left(\left(\frac{h}{L}\right)^2\left(\frac{t}{L}\right)^2\left(\frac{n\pi}{2}\right)^2+\frac{12k_s h(1-\nu^2)}{E}\left(\frac{t}{L}\right)^2\right)}{\left(\frac{n\pi}{2}\right)^2\left(\left(\frac{h}{L}\right)^2+\left(\frac{t}{L}\right)^2\right)+\frac{12k_s h(1-\nu^2)}{E}}, \tag{34}$$

for the dimensionless horizontal stress measure

$$\sigma^*=12(1-\nu^2)\frac{\sigma}{E}. \tag{35}$$

We consider two limit cases. For $k_s h \to \infty$ we obtain the Euler buckling stress for a beam or plate in plane strain, viz

$$\sigma=\frac{E}{12(1-\nu^2)}\left(n\pi\frac{t}{2L}\right)^2 \tag{36}$$

For $k_s h \to 0$ and/or $t/(nL) \to 0$ we obtain the buckling stress of the individual laminates, that is

$$\sigma=\frac{E}{12(1-\nu^2)}\frac{\left(n\pi\frac{h}{2L}\right)^2}{1+\left(\frac{h}{t}\right)^2} \tag{37}$$

The result (37) is not exact, since the term $(h/t)^2$ in the denominator includes only the first terms from the homogenisation of the original system. This is typical of most continuum approximations for layered systems, and is generally viewed in just this manner, as an approximation. However, the continuum can also be understood as a limiting case of zero lamination thickness. It is not possible to take such a limit in a straight forward manner, since divergence would result. However, one can imagine adjusting the microscopic moduli as the lamination thickness vanishes, in a manner such that the physically relevant ratios remain finite in the limit. The resulting ratios defines the continuum's physical parameters. After the limiting process is complete, the observed macroscopic moduli are used together with this ratio, to define an "effective" lamination thickness parameter h. When viewed in this way, the parameter h above is not the actual lamination thickness, but rather a "renormalised" parameter of the generalised continuum that depends upon scale, and in practice, is defined with reference to a set of observed macroscopic moduli. Nevertheless, it has a clear interpretation as a "lamination thickness parameter", and as such, behaves in a the manner one would expect of such a quantity.

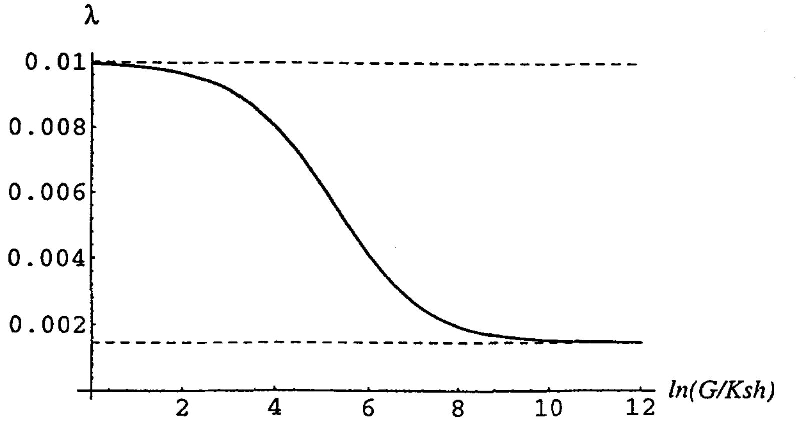

Figure 6. Critical dimensionless buckling load σ^* as a function of the (log of the) dimensionless shear modulus $\ln(G/k_s h)$, assuming $\nu = 0.2$, $t/L = 0.25$ and $h/t = 0.1$.

5. CONCLUSION

The crucial feature of the model presented is that it considers the influence of the bending stiffness of the individual layers within the framework of a continuum theory. The consideration of bending effects is crucial to preserving the uniqueness of the solutions upon yielding of the interface material. In the latter case the tangential stiffness vanishes, which in a standard model, would lead to singularity of the differential problem. The theory is a generalisation of the Cosserat and related continuum theories, and at the same a special case of some theories of a granular medium (corresponding to a highly anisotropic contact distribution). The formulation of constitutive relations which generate moments from the spin of the micro structure relative to the deformation spin, is simpler than the Cosserat theory would allow. It is also more satisfactory from a physical point of view since moment stresses vanish upon pure bending, which was not the case in the earlier Cosserat models.

Application to a simply supported, one-dimensional model for a deep, layered beam shows that the model accurately reproduces the results of Timoshenko's theory when the interface stiffness is large. (It being an exact match for infinite interface stiffness). Both the bending and buckling solutions demonstrate the theory's ability to interpolate smoothly between the known solutions for infinite interface stiffness and zero interface stiffness, while comparison of the bending results with finite element solutions, demonstrate an accurate representation of bending in the transition regime. The difference between the predictions of the present model and the much more complex finite element solution is less than 5% in the parameter range considered. This is despite the fact that the "h" appearing in the continuum theory need not coincide exactly with the actual lamination thickness.

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